

# Order-isomorphic Morass-definable $\eta_1$ -orderings

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## Abstract

We prove that in the Cohen extension adding  $\aleph_3$  generic reals to a model of  $ZFC + CH$  containing a simplified  $(\omega_1, 2)$ -morass, gap-2 morass-definable  $\eta_1$ -orderings with cardinality  $\aleph_3$  are order-isomorphic. Hence it is consistent that the  $2^{\aleph_0} = \aleph_3$  and that morass-definable  $\eta_1$ -orderings with cardinality of the continuum are order-isomorphic. We prove that there are ultrapowers of  $\mathbb{R}$  over  $\omega$  that are gap-2 morass-definable. The constructions use a simplified gap-2 morass, and commutativity with morass-maps and morass-embeddings, to extend a transfinite back-and-forth construction of order type  $\omega_1$ , to a function between objects of cardinality  $\aleph_3$ .

## 1 Introduction

Let  $\alpha$  be an ordinal. An  $\eta_\alpha$ -ordering without endpoints,  $\langle X, < \rangle$ , is a linear-ordering for which less than  $\aleph_\alpha$ -many consistent order constraints are necessarily witnessed by an object in  $X$ . That is, if  $L \subset X$  and  $U \subset X$  have cardinal less than  $\aleph_\alpha$ , and for every  $l \in L$  and  $u \in U$ ,  $l < u$ , then there is  $x \in X$  such that for every  $l \in L$  and  $u \in U$ ,  $l < x < u$ . We consider only  $\eta_\alpha$ -orderings without endpoints, that is  $U$  or  $L$  above may be empty.

We are primarily interested in  $\eta_1$ -orderings. By the Compactness Theorem,  $\eta_1$ -orderings exist in proliferation at all infinite cardinalities, and there are many well-studied examples of  $\eta_1$ -orderings that play a significant role in logic, topology and analysis. It is an early result of Model Theory that  $\eta_1$ -orderings having cardinality  $\aleph_1$  are order-isomorphic. This is proved with a classic back-and-forth construction of the isomorphism. The argument is by transfinite construction of length  $\omega_1$ , requiring that only countably many order-commitments need be satisfied at any step of the construction. Hence it is a consequence of the Continuum Hypothesis ( $CH$ ) that  $\eta_1$ -orderings having cardinality of the continuum,  $2^{\aleph_0}$ , are order-isomorphic.

We are particularly interested in  $\eta_1$ -orderings without endpoints bearing cardinality of the continuum. We seek to find useful conditions on  $\eta_1$ -orderings in which  $CH$  fails and  $\eta_1$ -orderings bearing cardinality of the continuum are order-isomorphic. In [3] we showed that in a Cohen extensions adding  $\aleph_2$ -generic reals to a model of  $ZFC + CH$  containing a simplified  $(\omega_1, 1)$ -morass, there is a level order-isomorphism between morass-definable  $\eta_1$ -orderings with cardinality of the continuum (Theorem 5.4, [3]). The simplified-morass plays a critical role in that we reduce the element-wise construction of a function between orderings having cardinality  $\aleph_2$  to a construction of length  $\omega_1$ .

In this paper, working in the Cohen extension adding generic reals indexed by  $\omega_3$  to a model of  $ZFC + CH$  containing a simplified  $(\omega_1, 2)$ -morass, we define a level order-isomorphism between gap-2 morass-definable  $\eta_1$ -orderings having cardinality  $2^{\aleph_0} = \aleph_3$ . Our strategy is as follows. Given a simplified gap-2 morass,  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ , we construct a level order-isomorphism,  $\bar{F}$ , on the vertices of the morass below  $\omega_1$  in a manner essentially identical to the construction of [3]. Then  $\bar{F}$  is a level morass-commutative (below  $\omega_1$ ) order-isomorphism between those elements of the  $\eta_1$ -orderings that are also in the generic extension adding reals indexed by  $\omega_1$ . We use the embeddings of  $\mathcal{F}$  to extend  $\bar{F}$  to a level order-isomorphism between the  $\eta_1$ -orderings having

cardinality of the continuum in a model of set theory with  $\aleph_3$ -generic reals.

In the next paper we show that morass-definable and gap-2 morass-definable  $\eta_1$ -ordered real-closed fields bearing cardinality of the continuum are isomorphic in the Cohen extensions adding  $\aleph_2$  and  $\aleph_3$  generic reals, respectively, by an  $\mathbb{R}$ -linear order-preserving isomorphism. The role of  $\eta_1$ -ordered real-closed fields in the subject of automatic continuity (the existence of discontinuous homomorphisms of  $C(X)$ , the algebra of continuous real-valued functions on an infinite compact Hausdorff space,  $X$ ) has been well-explored in [1], [2], [4], [5], [6] and [11]. We use the techniques of this paper to show that it is consistent that the continuum has cardinality  $\aleph_3$ , and that there exists a discontinuous homomorphism of  $C(X)$ , for any infinite compact Hausdorff space  $X$ .

## 2 Preliminaries

Let  $M$  be a model of  $ZFC + CH$  containing a simplified  $(\omega_1, 2)$ -morass,  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ . Let  $P$  be the poset  $Fn(\omega_3 \times \omega, 2)$ ,  $G$  be  $P$ -generic over  $M$  and  $M^P$  be the forcing language of the poset  $P$  in  $M$ . Let  $\mathbf{X} \in M[G]$  and  $\mathbf{Y} \in M[G]$  be morass-definable  $\eta_1$ -orderings below  $\omega_1$  (Definition 5.3 [3]), with sets of morass-defining sets of terms  $X$  and  $Y$  (resp.) in  $M^P$ . Let  $\bar{P} = Fn(\omega_1 \times \omega, 2)$ , and  $\bar{G} = G \cap \bar{P}$  be the factor of  $G$  that is  $\bar{P}$ -generic over  $M$ . So  $\bar{G}$  adds generic reals indexed by  $\omega_1$ . Let  $\bar{X}$  and  $\bar{Y}$  be the restrictions of  $X$  and  $Y$  respectively to  $M^{\bar{P}}$ , and  $\bar{\mathbf{X}}$  and  $\bar{\mathbf{Y}}$  be the restrictions of  $\mathbf{X}$  and  $\mathbf{Y}$  respectively to  $M[\bar{G}]$ . By the construction of Theorem 5.4 [3] there is a level, morass-commutative term function that in  $M[\bar{G}]$  is an order-isomorphism from  $\bar{X}$  to  $\bar{Y}$ . Since we are not guaranteed that every countable subset of  $\omega_3$  is in the image of a morass map from a countable vertex, we may not be able to complete the construction of the function we seek through extensions by morass-commutativity alone.

We will construct a function between sets of terms with cardinality  $\aleph_3$

in the forcing language,  $M^P$ , by applying the embeddings of a simplified gap-2 morass to a function on a set of terms in  $M^{\bar{P}}$  with cardinality  $\aleph_1$ . If  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  is a simplified  $(\omega_1, 2)$ -morass, then  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  is a simplified  $(\omega_2, 1)$ -morass. It is easy to verify that every ordinal of  $\omega_3$  is in the image of a morass map of  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  from a vertex below  $\omega_2$ , and hence that every countable subset of  $\omega_3$  is in the image of a single morass function of  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$ . If  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  had the property that every ordinal less than  $\omega_2$  were in the image of a morass map from countable vertex, then every element of  $\omega_3$  would be in the image of a morass map with countable vertex. Thereby every countable subset of  $\omega_3$  would be in the image of a single morass map from a countable vertex, and the construction of Theorem 5.4 [3] would suffice to prove that there is a level morass-commutative term map from  $X$  to  $Y$  that in  $M[G]$  is an order-isomorphism from  $\mathbf{X}$  to  $\mathbf{Y}$ . However, we are not assured that every ordinal is anticipated by a morass map from a countable vertex of  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$ . Instead we employ a simplified  $(\omega_1, 2)$ -morass to anticipate countable subsets of  $\omega_3$  by countable subsets of  $\omega_1$ , and thereby construct a function between sets of terms in the forcing language having cardinality  $\aleph_3$  with an inductive construction of length  $\omega_1$ .

### 3 The Simplified Gap-2 Morass

For  $\kappa$  a regular cardinal, we define a simplified  $(\kappa, 2)$ -morass as in Definition 1.3 [10].

**Definition 3.1** (*Simplified  $(\kappa, 2)$  – morass*) *The structure  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  is a simplified  $(\kappa, 2)$ -morass provided it has the following properties:*

1.  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  is a neat simplified  $(\kappa^+, 1)$ -morass [9].
2.  $\forall \alpha < \beta \leq \kappa$ ,  $\mathcal{F}_{\alpha\beta}$  is a family of embeddings (see page 172, [10]) from  $\langle \langle \varphi_\zeta \mid \zeta < \theta_\alpha \rangle, \langle \mathcal{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta_\alpha \rangle \rangle$  to  $\langle \langle \varphi_\zeta \mid \zeta < \theta_\beta \rangle, \langle \mathcal{G}_{\zeta\xi} \mid \zeta < \xi \leq \theta_\beta \rangle \rangle$ .
3.  $\forall \alpha < \beta < \kappa$  ( $|\mathcal{F}_{\alpha\beta}| < \kappa$ ).

4.  $\forall \alpha < \beta < \gamma \leq \kappa$  ( $\mathcal{F}_{\alpha\gamma} = \{f \circ g \mid f \in \mathcal{F}_{\beta\gamma}, g \in \mathcal{F}_{\alpha\beta}\}$ ). Here  $f \circ g$  is defined by:

$$(f \circ g)_\zeta = f_{g(\zeta)} \circ g_\zeta \quad \text{for } \zeta \leq \theta_\alpha,$$

$$(f \circ g)_{\zeta\xi} = f_{g(\zeta)g(\xi)} \circ g_{\zeta\xi} \quad \text{for } \zeta < \xi \leq \theta_\alpha.$$

5.  $\forall \alpha < \kappa$ ,  $\mathcal{F}_{\alpha, \alpha+1}$  is an amalgamation (see page 173 [10]).
6. If  $\beta_1, \beta_2 < \beta \leq \kappa$ ,  $\alpha$  a limit ordinal,  $f_1 \in \mathcal{F}_{\beta_1\alpha}$  and  $f_2 \in \mathcal{F}_{\beta_2, \alpha}$ , then  $\exists \beta(\beta_1, \beta_2 < \beta < \alpha$  and  $\exists f'_1 \in \mathcal{F}_{\beta_1\beta} \exists f'_2 \in \mathcal{F}_{\beta_2\beta} \exists \in \mathcal{F}_{\gamma\alpha}(f_1 = g \circ f'_1$  and  $f_2 = g \circ f'_2))$ .
7. If  $\alpha \leq \kappa$  and  $\alpha$  is a limit ordinal, then:

- (a)  $\theta_\alpha = \bigcup \{f[\theta_\beta] \mid \beta < \alpha, f \in \mathcal{F}_{\beta\alpha}\}$ .
- (b)  $\forall \zeta \leq \theta_\alpha$ ,  $\varphi_\zeta = \bigcup \{f_{\bar{\zeta}}[\varphi_{\bar{\zeta}}] \mid \exists \beta < \alpha (f \in \mathcal{F}_{\beta\alpha}, f(\bar{\zeta}) = \zeta)\}$ .
- (c)  $\forall \zeta < \xi \leq \theta_\alpha$ ,  $\mathcal{G}_{\zeta\xi} = \bigcup \{f_{\bar{\zeta}\bar{\xi}}[\mathcal{G}_{\bar{\zeta}\bar{\xi}}] \mid \exists \beta < \alpha (f \in \mathcal{F}_{\beta\alpha}, f(\bar{\zeta}) = \zeta, f(\bar{\xi}) = \xi)\}$ .

Suppose  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  is a simplified  $(\kappa^+, 1)$ -morass. It is routine to show that any subset of  $\kappa^{++}$  with cardinality less than  $\kappa^+$  is in the image of a single morass map,  $g \in \mathcal{G}_{\zeta\kappa^+}$ , for some  $\zeta < \kappa^+$ . We consider a somewhat stronger claim regarding subsets of  $\kappa^{++}$  with cardinality less than  $\kappa$ .

**Lemma 3.2** *Let  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  be a simplified  $(\kappa, 2)$ -morass, and  $T$  be a subset of  $\kappa^{++}$  with cardinality less than  $\kappa$ . Then there exists  $\beta < \kappa$ ,  $\bar{\zeta} < \theta_\beta$ ,  $\zeta < \kappa^+$ ,  $f \in \mathcal{F}_{\beta\kappa}$ ,  $g \in \mathcal{G}_{\zeta\kappa^+}$  and  $\bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_\beta}$  satisfying*

1.  $f(\bar{\zeta}) = \zeta$ .
2.  $f_{\bar{\zeta}\theta_\beta}(\bar{g}) = g$ .
3.  $T \subseteq g \circ f_{\bar{\zeta}}[\varphi_{\bar{\zeta}}] = f_{\bar{\zeta}\theta_\beta}(\bar{g}) \circ f_{\bar{\zeta}}[\varphi_{\bar{\zeta}}] = f_{\theta_\beta} \circ \bar{g}[\varphi_{\bar{\zeta}}]$ .

Proof: Let  $T$  be a countable subset of  $\kappa^{++}$ . Then there is a limit  $\zeta < \kappa^+$ ,  $S \subseteq \varphi_\zeta$  having cardinality less than  $\kappa$ , and  $g \in \mathcal{G}_{\zeta\kappa^+}$  such that  $g \upharpoonright_S: S \rightarrow T$  is a bijection. By Condition 7c of Definition 3.1, there is a  $\beta < \kappa$ ,  $f \in \mathcal{F}_{\beta\kappa}$ ,  $\bar{\zeta} \in \theta_\beta$  and  $\bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_\beta}$ , such that  $f(\bar{\zeta}) = \zeta$  and  $f_{\bar{\zeta}\theta_\beta}(\bar{g}) = g$ . By repeated applications of Property 5b of Definition 3.1, there is a (potentially larger)  $\beta < \kappa$ , and  $f \in \mathcal{F}_{\beta\kappa}$  with  $f(\bar{\zeta}) = \zeta$ , such that  $S \subseteq f_{\bar{\zeta}}[\varphi_{\bar{\zeta}}]$ . Let  $R = f_{\bar{\zeta}}^{-1}[S]$ . Then  $f_{\bar{\zeta}} \upharpoonright_R: R \rightarrow S$  is a bijection and

$$T \subseteq g \circ f_{\bar{\zeta}}[\phi_{\bar{\zeta}}].$$

□

Let  $P$  be the poset adding generic reals indexed by  $\kappa$ .

**Definition 3.3** ( $\lambda$ -support) *Let  $\kappa$  be a cardinal,  $\lambda < \kappa$  and  $P$  be the poset adding generic reals indexed by  $\kappa$ . Let  $\tau \in M^P$ . If there is  $S \subseteq \kappa$  with cardinality no greater than  $\lambda$  such that  $\tau \in M^{P(S)}$ , then  $\tau$  has  $\lambda$ -support. If  $T \subseteq M^P$  and every member of  $T$  has  $\lambda$ -support, then we say  $T$  has  $\lambda$ -support.*

**Corollary 3.4** *Let  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  be a simplified  $(\kappa, 2)$ -morass. If  $\tau \in M^P$  has  $\lambda$ -support for some  $\lambda < \kappa$ , then there is  $\zeta < \xi \leq \kappa^+$ ,  $g \in \mathcal{G}_{\zeta\xi}$ ,  $\beta < \kappa$ ,  $f \in \mathcal{F}_{\beta\kappa}$ ,  $\bar{\zeta} < \theta_\beta$ ,  $\bar{\xi} < \kappa$  and  $\bar{\tau} \in M^{P(\varphi_{\bar{\zeta}})}$  such that  $f(\bar{\zeta}) = \zeta$  and  $\tau = g \circ f_{\bar{\zeta}}(\bar{\tau})$ .*

Proof: Let  $S \subseteq \kappa$  have cardinality  $\lambda$  and  $\tau \in M^{P(S)}$ . By Lemma 3.2, there are  $\beta, \bar{\zeta}, \zeta, f$ , and  $g$  such that

$$S \subseteq g \circ f_{\bar{\zeta}}[\varphi_{\bar{\zeta}}].$$

Therefore there is  $\bar{\tau} \in M^{P(\varphi_{\bar{\zeta}})}$  such that

$$\tau = g \circ f_{\bar{\zeta}}(\bar{\tau}).$$

□

## 4 Continuous Extensions of Term Functions

When constructing an order-isomorphism between morass-definable  $\eta_1$ -orderings in the Cohen extension adding  $\aleph_2$ -generic reals [3], we showed that under suitable circumstances we were able to:

1. extend a term function that is forced to be an order-preserving injection by a single term of strict level and
2. extend a term function that is forced to be an order-preserving injection by commutativity with morass maps.

We will refer to these extensions as discrete and continuous, respectively. In this section we prove that we are able to continuously extend term functions by commutativity with the embeddings of a simplified gap-2 morass.

As in the proof that morass-definable  $\eta_1$ -orderings are order-isomorphic in the Cohen extension adding  $\aleph_2$ -generic reals, we need to continuously extend a back-and-forth construction with order-type  $\omega_1$  to a function between sets of higher cardinality by utilizing order-preserving injections on ordinals. As in [3], if  $\lambda$  and  $\nu$  are ordinals and  $\tau \in M^{P(\lambda)}$  (where  $P(\lambda) = Fn(\lambda \times \omega, 2)$ ), and  $h : \lambda \rightarrow \nu$  is an injection, we define  $h(\tau) \in M^{P(\nu)}$  as the term in the forcing language that results from the formal substitution of all instances of  $\lambda$  in  $\tau$  by their images under  $h$ . Given a term,  $\tau$ , in the forcing language  $M^{P(\bar{\zeta})}$ , for  $\bar{\zeta} < \kappa$ , and embedding  $f \in \mathcal{F}$ , we thereby specify a term in the forcing language adding Cohen reals indexed by  $\varphi_{f(\bar{\zeta})}$  by  $f_{\bar{\zeta}}(\tau)$ .

We adopt the notions of Morass-Commutativity and Morass-Definability from [3] to a higher cardinality gap-1 morass and a gap-2 morass.

**Definition 4.1** (*Morass-Commutative*) Let  $\kappa$  be regular,  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  be a simplified  $(\kappa, 1)$ -morass,  $\lambda \leq \kappa$  and  $X \subseteq M^{P(\kappa)}$ . We say that  $X$  is morass-commutative beneath  $\lambda$  provided that for any  $\zeta < \xi \leq \lambda$  and  $g \in \mathcal{G}_{\zeta\xi}$ ,  $x \in X \cap M^{P(\varphi_\zeta)}$  iff  $g(x) \in X$ . We say that  $X$  is morass-commutative if  $X$  is morass-commutative beneath  $\kappa$ .

**Definition 4.2** (*Morass-Definable*) Let  $\kappa$  be regular,  $M$  a c.t.m. of ZFC,  $P$  be the poset adding generic reals indexed by  $\kappa^+$ ,  $G$  be  $P$ -generic over  $M$ ,  $\mathbf{X} \in M[G]$  and  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  be a simplified  $(\kappa, 1)$ -morass. For  $\lambda < \kappa$  we say that  $\mathbf{X}$  is morass-definable below  $\lambda$  (with respect to  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$ ) provided that there is a discerning set of terms,  $X \subseteq M^P$ , morass-commutative beneath  $\lambda$ , such that the interpretation of  $X$  in  $M[G]$  is  $\mathbf{X}$ . We say that  $\mathbf{X}$  is morass definable if  $\mathbf{X}$  is morass-definable beneath  $\kappa$ .

**Definition 4.3** (*Morass-Continuation*) Let  $\kappa$  be a regular ordinal,  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  be a simplified  $(\kappa, 2)$ -morass and  $\bar{X} \subseteq M^{P(\kappa)}$  be a set of terms in the forcing language adding Cohen-generic reals indexed by  $\kappa$ . The morass continuation of  $\bar{X}$  is

$$X = \{f_{\theta_\beta} \circ \bar{g}(x) \mid x \in \bar{X} \cap M^{P(\varphi_{\bar{\zeta}})}, \beta < \kappa, f \in \mathcal{F}_{\beta\kappa}, \bar{\zeta} < \theta_\beta, \bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_\beta}\}.$$

We observe that

$$X = \{f_{\bar{\zeta}\theta_\beta}(\bar{g}) \circ f_{\bar{\zeta}} \mid x \in \bar{X} \cap M^{P(\varphi_{\bar{\zeta}})}, \beta < \kappa, f \in \mathcal{F}_{\beta\omega_1}, \bar{\zeta} < \theta_\beta, \bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_\beta}\}.$$

**Definition 4.4** (*Gap-2 Morass-Commutative*) Let  $\kappa$  be regular,  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  be a simplified  $(\kappa, 2)$ -morass,  $\lambda \leq \kappa^+$  and  $X \subseteq M^{P(\kappa)}$ . We say that  $X$  is gap-2 morass-commutative beneath  $\lambda$  provided that

1. For any  $\zeta < \xi \leq \lambda$ ,  $x \in M^{P(\varphi_\zeta)}$  and  $g \in \mathcal{G}_{\zeta\xi}$ ,  $x \in X$  iff  $g(x) \in X$ .
2. For any  $\beta < \omega_1$ ,  $f \in \mathcal{F}_{\theta_\beta\kappa}$  and  $\bar{\zeta} < \theta_\beta$  such that  $f(\bar{\zeta}) = \zeta \leq \lambda$ ,  $x \in X \cap M^{P(\varphi_{\bar{\zeta}})}$  iff  $f_{\bar{\zeta}}(x) \in X$ .

If  $X$  is gap-2 morass-commutative beneath  $\kappa^+$  we say it is gap-2 morass-commutative.

**Definition 4.5** (*Gap-2 Morass-Definable*) Assume



1.  $M$  is a c.t.m. of ZFC.
2.  $\kappa$  is regular.
3.  $P$  is the poset that adds  $\kappa^{++}$  Cohen generic reals.
4.  $G$  is  $P$ -generic over  $M$ .
5.  $\mathbf{X} \in M[G]$ .
6.  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  is a simplified  $(\kappa, 2)$ -morass.

We say that  $\mathbf{X}$  is gap-2 morass-definable (with respect to  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ ) provided that there is  $X \subset M^{P(\kappa)}$  satisfying

1.  $X$  is a set of discerning terms.
2.  $X$  is morass commutative beneath  $\kappa$ .
3. The interpretation in  $M[G]$  of the gap-2 morass-continuation of  $X$  (with respect to  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ ) is  $\mathbf{X}$ .

Then  $\mathbf{X}$  is gap-2 morass-definable, and is gap-2 morass-defined by  $X$  and  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ .

**Lemma 4.6** *If  $X$  is the gap-2 morass-continuation of  $\bar{X} \in M^{P(\kappa^+)}$ , and  $\bar{X}$  is morass-commutative beneath  $\kappa$ , then  $X$  is gap-2 morass-commutative.*

Proof: Let  $X$  and  $\bar{X}$  satisfy the hypotheses of the Lemma. Let  $\zeta < \xi \leq \kappa^+$ ,  $g \in \mathcal{G}_{\zeta\xi}$  and  $x \in X$  with support in  $\varphi_\zeta$ . Then there are  $\beta < \omega_1$ ,  $f \in \mathcal{F}_{\beta\omega_1}$ ,  $\bar{\zeta} < \theta_\beta$ ,  $\bar{\xi} \leq \kappa$  and  $\bar{g} \in \mathcal{G}_{\bar{\zeta}\bar{\xi}}$  such that  $f(\bar{\zeta}) = \zeta$ ,  $f(\bar{\xi}) = \xi$  and  $f_{\bar{\zeta}\bar{\xi}}(\bar{g}) = g$ . We may assume that  $\bar{\zeta}$  is chosen so that  $\text{support}(x) \subseteq f_{\bar{\zeta}}[\varphi_{\bar{\zeta}}]$ . Let  $\bar{x} \in \bar{X}$  have support in  $\varphi_{\bar{\zeta}}$  and  $f_{\bar{\zeta}}(\bar{x}) = x$ . Then

$$g(x) = g \circ f_{\bar{\zeta}}(\bar{x}) = f_{\bar{\xi}} \circ \bar{g}(\bar{x}).$$

However  $\bar{g}(\bar{x}) \in \bar{X}$ , so  $g(x)$  is in the gap-2 continuation of  $\bar{X}$ ,  $X$ .  $\square$

**Lemma 4.7** *Let  $X \subseteq M^P$  be a set of terms with  $< \kappa$ -support. If  $X$  is gap-2 morass-commutative, then  $X$  is morass-commutative.*

Proof: Assume that  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  is a simplified  $(\kappa, 2)$ -morass. Let  $X \in M^P$  be gap-2 morass-commutative with respect to  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ . Let  $\zeta < \xi \leq \kappa^+$ ,  $x \in X \cap M^{P(\phi_\zeta)}$  and  $g \in \mathcal{G}_{\zeta\xi}$ . We may assume that  $\kappa < \xi$ . Then there is  $\beta < \kappa$ ,  $f \in \mathcal{F}_{\theta_\beta \kappa^+}$ ,  $\bar{\zeta} < \bar{\xi} \leq \theta_\beta$  and  $\bar{g} \in \mathcal{G}_{\bar{\zeta}\bar{\xi}}$  such that  $f(\bar{\zeta}) = \zeta$ ,  $f(\bar{\xi}) = \xi$  and  $f_{\bar{\zeta}\bar{\xi}}(\bar{g}) = g$ . The support of  $x$  has cardinality no greater than  $\kappa$ , and  $x$  is in the morass-continuation of a set of terms contained in  $M^{P(\kappa^+)}$ . Hence we may choose  $\bar{\zeta} < \kappa$  and  $\bar{x} \in M^{P(\varphi_{\bar{\zeta}})}$  such that  $f_{\bar{\zeta}}(\bar{x}) = x$ . Since  $X$  is gap-2 morass-commutative,  $\bar{x} \in X$ . Then

$$g(x) = g \circ f_{\bar{\zeta}}(\bar{x}) = f_{\bar{\zeta}\bar{\xi}}(\bar{g}) \circ f_{\bar{\zeta}}(\bar{x}) = f_{\bar{\xi}} \circ \bar{g}(\bar{x}) \in X.$$

Conversely, suppose  $\zeta < \xi \leq \kappa^+$  and there is a term  $x \in M^{P(\varphi_\zeta)}$  and  $g \in \mathcal{G}_{\zeta\xi}$  such that  $g(x) \in X$ . then there is  $\beta < \kappa$ ,  $f \in \mathcal{F}_{\theta_\beta \kappa^+}$ ,  $\bar{\zeta} < \bar{\xi} \leq \theta_\beta$  and  $\bar{g} \in \mathcal{G}_{\bar{\zeta}\bar{\xi}}$  such that  $f(\bar{\zeta}) = \zeta$ ,  $f(\bar{\xi}) = \xi$  and  $f_{\bar{\zeta}\bar{\xi}}(\bar{g}) = g$ . Therefore there is  $\bar{x} \in M^{P(\varphi_{\bar{\zeta}})}$  such that  $x = f_{\bar{\zeta}}(\bar{x})$ . Then, since  $X$  is gap-2 morass-commutative,  $\bar{x} \in X$  and  $x \in X$ .  $\square$

We state the central theorem of this paper.

**Theorem 4.8** *Assume  $M$  is a c.t.m. of  $ZFC + CH$  containing a simplified  $(\omega_1, 2)$ -morass,  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ . Let  $P$  be the poset adding generic reals indexed by  $\omega_3$ . Assume  $X \in M^P$  and  $Y \in M^P$  are forced in all  $P$ -generic extensions to be  $\eta_1$ -orderings, and are gap-2 morass-defined (as orderings) by  $\bar{X} \subseteq M^{P(\omega_1)}$  and  $\bar{Y} \subset M^{P(\omega_1)}$  with respect to  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ . Then there is a level term function,  $\bar{F} : \bar{X} \rightarrow \bar{Y}$ , with gap-2 morass-continuation,  $F : X \rightarrow Y$ , that in any  $P$ -generic extension of  $M$ ,  $M[G]$ , defines an order-isomorphism from the interpretation of  $X$  in  $M[G]$  to the interpretation of  $Y$  in  $M[G]$ .*

Proof: Let  $P$ ,  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ ,  $X$ ,  $Y$ ,  $\bar{X}$  and  $\bar{Y}$  satisfy the hypotheses of the Theorem. For  $h$  an injection on  $S \subseteq \omega_2$ , and  $\tau \in M^P$  with support

contained in  $S$ , we define  $h(\tau)$  as in [3]. That is,  $h(\tau)$  is a term in the forcing language adding generic reals indexed by  $h[S]$  in which every indexing ordinal appearing in  $\tau$  is formally replaced by its image under  $h$ . In [3] we observed that given a simplified  $(\omega_1, 1)$ -morass, every countable subset of  $\omega_2$  is in the image of a single morass function from a countable vertex. Every countable subset of  $\omega_3$  is in the image of a single morass function  $g \in \mathcal{G}_{\zeta\omega_2}$  for some  $\zeta < \omega_2$ . In general  $\zeta$  need not be countable, so we are not able to use continuous extensions from countable vertices of  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  to complete the construction of a term function that in  $M[G]$  is an order-isomorphism.

The sets of terms  $\bar{X}$  and  $\bar{Y}$  satisfy the hypotheses of Theorem 5.4 of [3], along the initial  $\omega_1$ -segment of  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$ . Therefore there is a level, morass-commutative bijection (below the  $\omega_1$ ),  $\bar{F} : \bar{X} \rightarrow \bar{Y}$ , that in any  $P(\omega_1)$ -generic extension of  $M$  defines an order-isomorphism from the interpretation of  $\bar{X}$  to the interpretation of  $\bar{Y}$ . An  $(\omega_1, 2)$ -morass comes equipped with a set of embeddings of countable initial segments of an  $(\omega_2, 1)$ -morass. We use these embeddings to complete the construction of a level term function that is forced to be an order-isomorphism between the interpretation of  $X$  and the interpretation of  $Y$  in any  $P$ -generic extension of  $M$ . Unlike the arguments from [3] employing an  $(\omega_1, 1)$ -morass, we cannot fulfill the analogous requirement that every countable subset of  $\omega_3$  is in the image of a single morass map,  $g \in \mathcal{G}_{\zeta\omega_2}$  for  $\zeta < \omega_1$ . However, we are able to utilize the embeddings of  $\mathcal{F}$  to generate all countable subsets of  $\omega_3$  as images of compositions of embeddings of  $\mathcal{F}$  with morass maps from countable vertices of  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$ . Let

$$H = \{f_{\theta_\beta} \circ \bar{g} \mid \beta < \omega_1, f \in \mathcal{F}_{\beta\omega_1}, \bar{\zeta} < \theta_\beta, f(\bar{\zeta}) = \zeta, \bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_\beta}\}.$$

Since  $\mathcal{F}$  is an embedding, we have that

$$H = \{f_{\bar{\zeta}\theta_\beta}(\bar{g}) \circ f_{\bar{\zeta}} \mid \beta < \omega_1, f \in \mathcal{F}_{\beta\omega_1}, \bar{\zeta} < \theta_\beta, f(\bar{\zeta}) = \zeta, \bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_\beta}\}.$$

By Lemma 3.2, any countable subset of  $\omega_3$  is a subset of the range of a single  $h \in H$ . Let  $F$  be the gap-2 morass-continuation of  $\bar{F}$ . By Lemma 4.6,  $F$  is

gap-2 morass-commutative. By Lemma 4.7,  $F$  is morass-commutative.  $F$  is a relation on  $X \times Y$ . Since  $\bar{F}$  is level,  $F$  is level. We show in any  $P$ -generic extension of  $M$ ,  $M[G]$ ,  $F : X \rightarrow Y$  is forced to be an order-isomorphism.

Let  $\tau = (x, y) \in F$ . So there is  $\beta < \omega_1$ ,  $f \in \mathcal{F}_{\beta\omega_1}$ ,  $\zeta < \omega_2$ ,  $\bar{\zeta} < \theta_\beta$ ,  $\bar{\tau} \in \bar{F} \cap M^{P(\varphi_{\bar{\zeta}})}$ , and  $g \in \mathcal{G}_{\zeta\omega_2}$  such that  $\tau = g \circ f_{\bar{\zeta}}(\bar{\tau})$ . Since  $\mathcal{F}$  is an embedding, there is  $\bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_\beta}$  where

$$f_{\theta_\beta} \circ \bar{g} = f_{\bar{\zeta}\theta_\beta}(\bar{g}) \circ f_{\bar{\zeta}}.$$

Suppose there are  $\beta' < \omega_1$ ,  $f' \in \mathcal{F}_{\beta'\omega_1}$ ,  $\zeta' < \omega_2$ ,  $\bar{\zeta}' < \theta_{\beta'}$ ,  $g' \in \mathcal{G}_{\zeta'\omega_2}$ ,  $\tau' = (x, y') \in F$  and  $\bar{\tau}' \in \bar{F} \cap M^{P(\varphi_{\bar{\zeta}'})}$  such that  $\tau' = g' \circ f_{\bar{\zeta}'}(\bar{\tau}')$ . In order to show that  $F$  is a function on terms, it is sufficient to show that  $y = y'$ . Since  $\mathcal{F}$  is an embedding, there is  $\bar{g}' \in \mathcal{G}_{\bar{\zeta}'\theta_{\beta'}}$  such that

$$f'_{\theta_{\beta'}} \circ \bar{g}' = f'_{\bar{\zeta}'\theta_{\beta'}}(\bar{g}') \circ f'_{\bar{\zeta}'}$$

We may assume without loss of generality that  $\zeta' > \zeta$ . We observe first that since  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$  is a simplified gap-1 morass, there are  $g'_1 \in \mathcal{G}_{\zeta_1\zeta_2}$  and  $g \in \mathcal{G}_{\zeta_2\omega_2}$  such that

$$g_1 = g \circ g'_1.$$

Then

$$g \circ (f_1)_{\bar{\zeta}_1}(\bar{\tau}_1)_x = (f_2)_{\bar{\zeta}_2}(\bar{\tau}_2)_x.$$

Therefore  $g$  and  $g_2$  are equal on the support of  $(f_1)_{\bar{\zeta}_1}((\bar{\tau}_1)_x)$ , and we may assume that  $g = g_2$ . The gap-2 continuation of a gap-2 morass-commutative term function is a term function and

$$g'_1 \circ (f_1)_{\bar{\zeta}_1}((\bar{\tau}_1)_y) = (f_2)_{\bar{\zeta}_2}((\bar{\tau}_2)_y).$$

By condition 6 of Definition 3.1, there is  $\beta < \omega_1$ ,  $\beta_1, \beta_2 < \beta$ ,  $(f_1)' \in \mathcal{F}_{\beta_1\beta}$ ,  $(f_2)' \in \mathcal{F}_{\beta_2\beta}$  and  $f \in \mathcal{F}_{\beta\omega_1}$  such that  $f_1 = f \circ (f_1)'$  and  $f_2 = f \circ (f_2)'$ . Hence we may simplify the situation as follows. We assume that

1.  $\beta < \omega_1$ .

2.  $\omega_1 < \zeta < \zeta' < \omega_2$ .
3.  $h \in F$ ,  $h' \in F$ ,  $h(x) = y$  and  $h'(x) = y'$ .
4.  $f \in \mathcal{F}_{\beta\omega_1}$ ,  $f(\bar{\zeta}) = \zeta$  and  $f(\bar{\zeta}') = \zeta'$ .
5.  $h = f_{\theta_\beta} \circ \bar{g}$ ,  $\bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_\beta}$  and  $g = f_{\bar{\zeta}\zeta}(\bar{g})$ .
6.  $h' = f_{\theta_\beta} \circ \bar{g}'$ ,  $\bar{g}' \in \mathcal{G}_{\bar{\zeta}'\theta_\beta}$  and  $g = f_{\bar{\zeta}'\zeta'}(\bar{g}')$ .

We show that  $y' = y$ . We observe that since  $f$  is a gap-2 morass-embedding,

$$h = f_{\theta_\beta} \circ \bar{g} = f_{\bar{\zeta}\zeta}(\bar{g}) \circ f_{\bar{\zeta}}$$

and

$$h' = f_{\theta_\beta} \circ \bar{g}' = f_{\bar{\zeta}'\zeta'}(\bar{g}') \circ f_{\bar{\zeta}'}$$

The support of  $x$  is in the range of  $g$  and  $g'$ , and is hence in the range of  $f_{\theta_\beta}$ . Let  $x_{\theta_\beta} = f_{\theta_\beta}^{-1}(x)$ . The support of  $x_{\theta_\beta}$  is in the ranges of  $\bar{g}$  and  $\bar{g}'$ . Let  $x_{\bar{\zeta}} = \bar{g}^{-1}(x_{\theta_\beta})$  and  $x_{\bar{\zeta}'} = (\bar{g}')^{-1}(x_{\theta_\beta})$ .

By condition (P4) of Definition 1.1 [10], there are  $g^* \in \mathcal{G}_{\bar{\zeta}'\bar{\zeta}}$  and  $g^\sharp \in \mathcal{G}_{\bar{\zeta}\theta_\beta}$  such that  $\bar{g}' = g^\sharp \circ g^*$ . By Lemma 1.4, [10],  $g^\sharp$  and  $\bar{g}$  are equal on the support of  $x_{\bar{\zeta}}$ . Hence we assume that  $g^\sharp = \bar{g}$ ,  $\bar{g}'(x_{\bar{\zeta}'}) = \bar{g} \circ g^*(x_{\bar{\zeta}})$  and  $g^*(x_{\bar{\zeta}}) = x_{\bar{\zeta}}$ . The range of  $f_{\theta_\beta}$  contains the support of  $x$ , and since  $h$  and  $h'$  are level term functions, the range of  $f_{\theta_\beta}$  also contains the supports of  $y$  and  $y'$ . Let  $y_{\theta_\beta} = f_{\theta_\beta}^{-1}(y)$  and  $y'_{\theta_\beta} = f_{\theta_\beta}^{-1}(y')$ . Then  $y = y'$  if and only if  $y_{\theta_\beta} = y'_{\theta_\beta}$ . Similarly, let  $y_{\bar{\zeta}} = \bar{g}^{-1}(y_{\theta_\beta})$  and  $y'_{\bar{\zeta}} = \bar{g}^{-1}(y'_{\theta_\beta})$ . Then since  $\bar{F}$  is a morass-commutative term function below  $\omega_1$ ,  $\bar{F}(x_{\bar{\zeta}}) = y_{\bar{\zeta}} = y'_{\bar{\zeta}}$ . It follows that  $y = y'$ , and  $F$  is a well-defined term function.

By definition,  $X$  and  $Y$  are the gap-2 morass-continuations of  $\bar{X}$  and  $\bar{Y}$ , respectively. Therefore  $F : X \rightarrow Y$  is a bijection, and is forced to be a bijection in any  $P$ -generic extension of  $M$ .

Finally we show that  $F$  is order-preserving in  $M[G]$ . Suppose that  $x, x' \in X$  and it is forced in all generic extensions that  $x < x'$ . Let  $y, y' \in T_Y$  and  $F(x) = y$  and  $F(x') = y'$ . By the arguments above, there are

1.  $\beta < \omega_1$
2.  $f \in \mathcal{F}_{\beta\omega_1}$
3.  $\bar{\zeta} < \omega_1$
4.  $x_{\bar{\zeta}}, x'_{\bar{\zeta}}, y_{\bar{\zeta}}$  and  $y'_{\bar{\zeta}}$ , terms in  $M^{P(\varphi_{\bar{\zeta}})}$
5.  $\bar{g} \in \mathcal{G}_{\bar{\zeta}\theta_{\beta}}$ .

such that

1.  $f_{\theta_{\beta}} \circ \bar{g}((x_{\bar{\zeta}}, y_{\bar{\zeta}})) = (x, y)$
2.  $f_{\theta_{\beta}} \circ \bar{g}((x'_{\bar{\zeta}}, y'_{\bar{\zeta}})) = (x', y')$

Since the order relations  $<_X$  and  $<_Y$  are gap-2 morass-commutative it is forced in all  $P$ -generic extensions of  $M$  that  $x < x'$  just in case  $x_{\bar{\zeta}} < x'_{\bar{\zeta}}$  in all  $P(\varphi_{\bar{\zeta}})$ -generic extensions. Since  $F$  is the morass-continuation of  $\bar{F}$ ,  $\bar{F}(x_{\bar{\zeta}}) = y_{\bar{\zeta}}$  and  $\bar{F}(x'_{\bar{\zeta}}) = y'_{\bar{\zeta}}$ . The term function  $\bar{F}$  is forced to be order preserving. Hence it is forced in all  $P(\varphi_{\bar{\zeta}})$ -generic extensions that  $y_{\bar{\zeta}} < y'_{\bar{\zeta}}$ . By gap-2 morass-commutativity of  $<_Y$  it is forced that  $y < y'$ .  $\square$

This theorem generalizes to higher cardinalities in the obvious way.

**Corollary 4.9** *Let  $\kappa = \aleph_{\alpha}$  be regular. Assume  $M$  is a c.t.m. of  $ZFC + CH$  containing a simplified  $(\kappa, 2)$ -morass,  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ . Let  $P$  be the poset adding generic reals indexed by  $\kappa^{++}$ . Assume  $X \in M^P$  and  $Y \in M^P$  are forced in all  $P$ -generic extensions to be  $\eta_{\alpha}$ -orderings, and are gap-2 morass-defined (as orderings) by  $\bar{X} \subseteq M^{P(\omega_{\alpha})}$  and  $\bar{Y} \subset M^{P(\omega_{\alpha})}$  with respect to  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ . Then there is a level term function,  $\bar{F} : \bar{X} \rightarrow \bar{Y}$ , with gap-2 morass-continuation,  $F : X \rightarrow Y$ , that in any  $P$ -generic extension of  $M$ ,  $M[G]$ , defines an order-isomorphism from the interpretation of  $X$  in  $M[G]$  to the interpretation of  $Y$  in  $M[G]$ .*

## 5 Ultrapowers of $\mathbb{R}$ over $\omega$

We turn our attention to  $\mathbb{R}^\omega/U$ , an ultrapower of  $\mathbb{R}$  over a non-principal ultrafilter on  $\omega$ ,  $U$ . By results of G. Dales [1], J. Esterle [5] and B. Johnson [8] the existence of an  $\mathbb{R}$ -linear order-preserving monomorphism from the finite elements of  $\mathbb{R}^\omega/U$ , for some non-principal ultrafilter on  $\omega$ ,  $U$ , into the Esterle algebra is sufficient to prove the existence of a discontinuous homomorphism of  $C(X)$ , the algebra of continuous real valued functions on  $X$ , where  $X$  is an infinite compact Hausdorff space. In a later paper, relying essentially on the techniques of this paper, we prove that it is consistent that such a monomorphism exists in a model of set theory in which  $2^{\aleph_0} = \aleph_3$ . In anticipation of such a construction, we finish this paper with a proof that in the Cohen extension adding  $\aleph_3$ -generic reals of a model of  $ZFC+CH$  containing a simplified  $(\omega_1, 2)$ -morass, ultrapowers of  $\mathbb{R}$  over standard ultrafilters on  $\omega$  are gap-2 morass definable, and hence are order-isomorphic with other gap-2 morass-definable  $\eta_1$ -orderings. This result extends Theorem 6.15 [3], that in the Cohen extension adding  $\aleph_2$ -generic reals of a model of  $ZFC + CH$  containing a simplified gap-1 morass, ultrapowers of  $\mathbb{R}$  over standard ultrafilters on  $\omega$  are order-isomorphic.

By Theorem 4.8, in order to prove that an ultrapower of  $\mathbb{R}$ ,  $\mathbb{R}^\omega/U$ , is order-isomorphic with a gap-2 morass-definable  $\eta_1$ -ordering, it is sufficient to prove that  $\mathbb{R}^\omega/U$  is gap-2 morass-definable. We show that this is so, provided that  $U$  is a standard non-principal ultrafilter. We adapt the definition of standard ultrafilter (Definition 6.14 [3]) to a simplified gap-2 morass.

**Definition 5.1** (*Standard Term for a subset of  $\omega$* ) A standard term for a subset of  $\omega$  is a term  $x \in M^P$ , such that for each  $(\tau, p) \in x$ ,  $\tau$  is a canonical term in  $M^P$  for a natural number.

**Definition 5.2** (*Standard Term for an Ultrafilter*) Let  $\lambda \leq \omega_3$  and  $U \in M^{P(\lambda)}$  be a morass-closed set of standard terms for subsets of  $\omega$  (below  $\lambda$ )

such that for all  $\alpha \leq \lambda$ ,  $U \cap M^{P(\alpha)}$  is forced to be an ultrafilter in all  $P(\alpha)$ -generic extensions for  $M$ . Then  $U$  is a standard term for an ultrafilter below  $\lambda$ .

If  $U$  is a standard term for an ultrafilter below  $\omega_3$ , we say it is a standard ultrafilter.

**Definition 5.3** (*Complete Standard Term for an Ultrafilter*) Let  $U$  be a standard term for an ultrafilter below  $\lambda$ .  $U$  is complete provided that for every standard term for a subset of  $\omega$ ,  $u \in M^{P(\lambda)}$ ,  $u \in U$  iff  $\Vdash u \in U$ .

That is, below  $\lambda$ , every standard term for a subset of  $\omega$  that is forced to be in  $U$ , is a member of  $U$ . Every standard term for an ultrafilter has a complete extension. Let  $U$  be a complete standard term for an ultrafilter and  $u$  be a standard term for a subset of  $\omega$ . There is a standard term for a subset of  $\omega$  that decides the membership of  $u$  in  $U$  in all  $P(\lambda)$ -generic extensions. Since  $U$  is forced to be an ultrafilter,  $\{p \in P(\lambda) \mid p \Vdash u \in U \vee p \Vdash u \notin U\}$  is dense in  $P(\lambda)$ . We consider  $u$  as a term for a binary sequence. In this sense,  $p \Vdash n \in u$  iff  $p \Vdash u_n = 1$ . It is clear that there is a standard term for a binary sequence,  $v$ , such that  $p \Vdash u_n = 1$  iff  $p \Vdash v_n = 0$ . Let  $d(u)$  be the standard term for a subset of  $\omega$  so that it is forced that  $d(u) = u$  if  $p \Vdash u \in U$  and  $d(u) = v$  if  $p \Vdash u \notin U$ . Then it is forced in all generic extensions that  $d(u) \in U$ .

**Theorem 5.4** Let  $\kappa$  be regular,  $M$  be a c.t.m. of ZFC and  $P$  be the poset adding generic reals indexed by  $\kappa^{++}$ . Assume  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$  is a simplified  $(\kappa, 2)$ -morass in  $M$ . Let  $\bar{U}$  be a standard ultrafilter that is morass-commutative beneath  $\kappa$  with respect to  $\langle \vec{\varphi}, \vec{\mathcal{G}} \rangle$ , and  $U$  be the gap-2 continuation of  $\bar{U}$ . Then  $U$  is a standard gap-2 morass-commutative and morass-commutative ultrafilter.

Proof: Let  $\kappa$ ,  $M$ ,  $P$ ,  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ ,  $\bar{U}$  and  $U$  satisfy the hypotheses of the Theorem. We claim that in any generic extension of  $M$ ,  $U$  is closed under finite intersections, has f.i.p. and is closed under supersets.



Let  $u, v \in U$ . There is  $\zeta < \kappa^+$ ,  $g \in \mathcal{G}_{\zeta\kappa^+}$ ,  $\beta < \kappa$ ,  $f \in \mathcal{F}_{\beta\kappa}$ ,  $\bar{\zeta} < \theta_\beta$  and  $u_{\bar{\zeta}}, v_{\bar{\zeta}} \in M^{P(\varphi_{\bar{\zeta}})} \cap \bar{U}$  such that  $f(\bar{\zeta}) = \zeta$ ,  $g \circ f_{\bar{\zeta}}(u_{\bar{\zeta}}) = u$  and  $g \circ f_{\bar{\zeta}}(v_{\bar{\zeta}}) = v$ . It is forced that  $u_{\bar{\zeta}} \cap v_{\bar{\zeta}} \in \bar{U}$ . Hence it is forced that  $u \cap v \in U$ .

Let  $u, v \in U$ . Let  $\zeta < \kappa^+$ ,  $u_\zeta, v_\zeta \in U \cap M^{P(\varphi_\zeta)}$  and  $g \in \mathcal{G}$  be such that  $u = g(u_\zeta)$  and  $v = g(v_\zeta)$ . Let  $\beta < \kappa$ ,  $f \in \mathcal{F}_{\beta\kappa}$ ,  $\bar{\zeta} < \theta_\beta$  be such that  $f_{\bar{\zeta}}(u_{\bar{\zeta}}) = u_\zeta$  and  $f_{\bar{\zeta}}(v_{\bar{\zeta}}) = v_\zeta$ . It is forced in all generic extensions that  $u_{\bar{\zeta}} \cap v_{\bar{\zeta}} \neq \emptyset$ . The term function  $g \circ f_{\bar{\zeta}}$  is an order preserving injection on  $\varphi_{\bar{\zeta}}$ , so it is forced that  $u \cap v$  is non-empty. Hence  $U$  is forced to have f.i.p.

Let  $u$  and  $v$  be standard terms for subsets of  $\omega$ ,  $u \in U$  and  $\Vdash u \subseteq v$ . Let  $S$  be the union of the supports of  $u$  and  $v$ . There are  $\beta < \kappa$ ,  $f \in \mathcal{F}_{\beta\kappa}$ ,  $\bar{\zeta} < \theta_\beta$ ,  $\zeta < \kappa^+$  and  $g \in \mathcal{G}_{\zeta\kappa^+}$  such that

$$S \subseteq g \circ f[\varphi_{\bar{\zeta}}].$$

and  $\bar{u} \in \bar{U} \cap M^{P(\varphi_{\bar{\zeta}})}$  satisfying

$$g \circ f_{\bar{\zeta}}(\bar{u}) = u.$$

Let  $\bar{v} \in M^{P(\varphi_{\bar{\zeta}})}$  be such that  $f_{\bar{\zeta}}(\bar{v}) = v$ . Then  $\Vdash \bar{u} \subseteq \bar{v}$ . Since  $\bar{U}$  is forced to be an ultrafilter in all  $P(\kappa)$ -generic extensions of  $M$ ,  $\Vdash \bar{v} \in \bar{U}$ . Hence  $\Vdash v \in U$ . So it is forced that  $U$  is closed under supersets and is forced to be a filter.

We show that  $U$  is forced to be an ultrafilter. Let  $G$  be  $P$ -generic over  $M$ . Let  $\mathbf{u}$  be a subset of  $\omega$  in  $M[G]$ . Let  $u$  be a standard term for  $\mathbf{u}$ . Then there is  $\zeta < \kappa^+$ ,  $g \in \mathcal{G}_{\zeta\kappa^+}$  and a standard term  $u_\zeta \in M^{P(\varphi_\zeta)}$  such that  $g(u_\zeta) = u$ . Let  $\beta < \kappa$ ,  $f \in \mathcal{F}_{\beta\kappa}$ ,  $\bar{\zeta} < \theta_\beta$  and  $\bar{u}$  be a standard term for a subset of  $\omega$  with support contained in  $\varphi_{\bar{\zeta}}$  such that  $f_{\bar{\zeta}}(\bar{u}) = u_\zeta$ . Then  $\Vdash d(\bar{u}) \in \bar{U}$ . It follows that  $g \circ f_{\bar{\zeta}}(d(\bar{u})) \in U$ , and the membership of  $\mathbf{u}$  in  $U$  is decided. Hence  $U$  is forced to be an ultrafilter.

By Lemma 4.6,  $U$  is gap-2 morass-commutative. By Lemma 4.7,  $U$  is morass-commutative. The arguments above hold for all vertices below  $\kappa^+$  on  $\langle \vec{\varphi}, \vec{\mathcal{G}}, \vec{\theta}, \vec{\mathcal{F}} \rangle$ , so  $U$  is a standard ultrafilter.  $\square$

**Corollary 5.5** *Let  $M$  be a model of ZFC containing a simplified  $(\omega_1, 2)$ -morass. Let  $U_0$  be a non-principal ultrafilter on  $\omega$  in the ground model, and  $P$  be the poset adding  $\aleph_3$ -generic reals. Then there is  $\bar{U} \subseteq M^{P(\omega_1)}$ , a morass-commutative standard term for an ultrafilter below  $\omega_1$ , that is forced to extend  $U_0$ . The gap-2 continuation of  $\bar{U}$ ,  $U \in M^P$ , is a standard ultrafilter that is gap-2 morass-commutative and morass-commutative.*

Proof: Let  $U_0$  be an ultrafilter of the ground model. It is a consequence of Theorem 6.7 [3] that there is  $\bar{U}$  extending  $U_0$  that is a morass-commutative standard term for an ultrafilter below  $\omega_1$ . By Theorem 5.4, the gap-2 continuation of  $\bar{U}$ ,  $U$ , satisfies the Theorem.  $\square$

We pass now to consideration of ultrapowers of  $\mathbb{R}$  over  $\omega$ . Suppose that  $U$  is the gap-2 continuation of a complete standard term for an ultrafilter,  $\bar{U}$ , that is morass commutative beneath  $\omega_1$ . Then  $\mathbb{R}^\omega/U$  is the gap-2 continuation of the morass-closure (below  $\omega_1$ ) of  $\mathbb{R}^\omega/\bar{U}$  (the latter  $\mathbb{R}^\omega$  should be understood as the restriction to sequences of reals in  $M^{P(\omega_1)}$ ).

**Theorem 5.6** *Let  $\bar{U}$  be a standard term for an ultrafilter that is morass-commutative below  $\omega_1$ , and  $U$  be the gap-2 continuation of  $\bar{U}$ . Then  $\mathbb{R}^\omega/U$  is a gap-2 morass-definable  $\eta_1$ -ordering.*

Proof. Let  $\bar{U} \subseteq M^{P(\omega_1)}$  be a morass-commutative complete standard term for an ultrafilter (below  $\omega_1$ ) that has  $U \subseteq M^P$  as its gap-2 continuation. Every element of  $U$  has countable support, so by Lemma 3.2,  $U$  is a complete standard term for an ultrafilter. By Lemma 4.6 and Lemma 4.7,  $U$  is gap-2 morass-commutative and morass-commutative. Let  $\bar{X}$  be the morass-closure beneath  $\omega_1$  of an ultrapower generator in  $M^{P(\omega_1)}$  with respect to  $\bar{U}$  (see Definition 6.11 [3]). By Lemma 6.13 [3],  $\bar{X}$ , with the ordering imposed by  $\bar{U}$ , is an  $\eta_1$ -ordering that is morass-commutative beneath  $\omega_1$ . Let  $X$  be the gap-2 morass-continuation of  $\bar{X}$ . We consider  $X$  as a gap-2 morass-commutative set of discerning sequences of reals, with order imposed by  $U$ . We show that  $X$  is forced to be order-isomorphic with  $\mathbb{R}^\omega/U$ . Let  $\pi : X \rightarrow \mathbb{R}^\omega/U$

be the natural quotient map. Since  $\pi \upharpoonright_{\bar{X}}$  is an injection, so is  $\pi$ . We claim that  $\pi$  is a surjection. Let  $x$  be a discerning term for a sequence of real numbers. Then  $[x] \in \mathbb{R}^\omega/U$ . Since  $x$  has countable support, there is  $\zeta < \omega_1$  such that  $x \in M^{P(\varphi_\zeta)}$ . By Lemma 3.2,  $x = f_{\bar{\zeta}}(\bar{x})$ , for some  $\bar{\zeta} < \zeta$  and  $\bar{x} \in M^{P(\varphi_{\bar{\zeta}})}$ . Since  $\bar{U}$  is a standard term for an ultrafilter, it is forced in all generic extensions that  $[\bar{x}] \in \mathbb{R}^\omega/\bar{U}$ . The gap-2 morass-continuation of  $\bar{U}$ ,  $U$ , is morass-commutative, and hence it is forced that  $[x] \in \mathbb{R}^\omega/U$ , and  $\pi$  is forced to be a surjection. It is routine to verify that  $\pi$  is order-preserving, and hence is forced to be an order-isomorphism. Therefore  $X$  is a gap-2 morass-defined  $\eta_1$ -ordering.  $\square$

We get the following immediate consequences.

**Corollary 5.7** *If  $U_0$  is a non-principal ultrafilter in the ground model, then there is a standard, gap-2 morass-definable ultrafilter extending  $U_0$ ,  $U$ . Hence  $\mathbb{R}^\omega/U$  is a gap-2 morass-definable  $\eta_1$ -ordering.*

**Corollary 5.8** *Let  $U_0$  be a non-principal ultrafilter in the ground model and  $\langle Y, < \rangle$  be a gap-2 morass-definable. Then there is a standard, gap-2 morass-definable ultrafilter,  $U$ , such that  $\mathbb{R}^\omega/U$  is order-isomorphic with  $\langle Y, < \rangle$ .*

## 6 Next Results

In this paper and [3] we have extended the classical result that  $\eta_1$ -orderings of cardinality  $\aleph_1$  are order-isomorphic. In the next paper we extend the result that there is an  $\mathbb{R}$ -linear isomorphism between  $\eta_1$ -ordered real-closed fields of cardinality  $\aleph_1$ . We show that in the Cohen extension adding  $\aleph_2$ -generic reals to a model of ZFC+CH containing a simplified  $(\omega_1, 1)$ -morass, there is a morass-definable  $\mathbb{R}$ -linear isomorphism between  $\eta_1$ -ordered morass-definable real-closed fields. We then show that in the Cohen extension adding  $\aleph_3$ -generic reals to a model of ZFC+CH containing a simplified  $(\omega_1, 2)$ -morass, there is a gap-2 morass-definable  $\mathbb{R}$ -linear isomorphism between gap-2 morass-definable  $\eta_1$ -orderings. With these results in hand we are able to

extend the theorem of Woodin [11] that it is consistent that  $2^{\aleph_0} = \aleph_2$  and there exists a discontinuous homomorphism of  $C(X)$ . We show that it is consistent that  $2^{\aleph_0} = \aleph_3$  and that there exists a discontinuous homomorphism of  $C(X)$ , for any infinite compact Hausdorff space,  $X$ .

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